Some aspects of calculus on non-smooth sets

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Let E be a closed set in $\mathbf{R}^n$, and suppose that there is a $k \ge 1$ such that every $x, y \in E$ can be connected by a rectifiable path in E with length $\le k\,|x-y|$. This condition is satisfied by chord-arc curves, Lipschitz manifolds of any dimension, and fractals like Sierpinski gaskets and carpets. Note that length-minimizing paths in E are chord-arc curves with constant k.

A basic feature of this condition is that one can integrate local Lipschitz conditions on E to get global conditions. For instance, if $f : E \to \mathbf{R}$ is locally Lipschitz of order 1 with constant $C \ge 0$, then f is globally Lipschitz on E with constant $k\,C$.

Let $A(x)$ be a continuous function on E with values in linear mappings from $\mathbf{R}^n$ to $\mathbf{R}$. Equivalently, one can use the standard inner product on $\mathbf{R}^n$ to represent $A(x)$ by a continuous mapping from E into $\mathbf{R}^n$. Also let f be a locally Lipschitz real-valued function on E.

Suppose that A includes the directional derivatives of f almost everywhere on any rectifiable curve in E, in the sense that the derivative of $f(p(t))$ is equal to $A(p(t))$ applied to the derivative of $p(t)$ for almost every t when $p(t)$ is a locally Lipschitz function on an interval I in the real line with values in E. In particular, this holds when f is continuously differentiable on E in the sense of the Whitney extension theorem with differential at $x \in E$ given by $A(x)$. In this case,

$$f(p(b)) - f(p(a)) = \int_a^b A(p(t))(p'(t))\,dt \tag{1}$$

for every $a, b \in I$. If $A(x)$ is constant on E, then it follows that f is equal to the restriction of an affine function on $\mathbf{R}^n$ to E.

If A is not constant, then we still have that

$$\begin{aligned} &|f(y) - f(x) - A(x)(y-x)| \\ &\qquad \le k\,|x-y|\,\sup\{|A(w) - A(x)| : w \in E, |x-w| \le k\,|x-y|\} \end{aligned} \tag{2}$$

for every $x, y \in E$. This follows from the previous formula applied to a curve in E that connects x to y and has length $\le k\,|x-y|$, by approximating $A(p(t))$ in the integral by $A(x)$ and estimating the remainder by the oscillation of A on the path times the length of the path. If we did not know already that f is continuously differentiable on E in Whitney's sense, then it follows from this and the continuity of A. Even if we did know that f is continuously-differentiable

on E, this estimate provides more precise information about the behavior of f in terms of the continuity properties of A. For example, Hölder continuity of order α of A implies $C^{1,\alpha}$-type smoothness of f.

If E is a Lipschitz submanifold of $\mathbf{R}^n$, then E has tangent planes almost everywhere and a locally Lipschitz function f on E has linear differentials on almost all of these tangent planes. If there is a continuous function $A(x)$ on E with values in linear functionals on $\mathbf{R}^n$ such that these differentials of f are equal to the restriction of $A(x)$ to the tangent plane to E at almost everywhere $x \in E$, then similar reasoning shows that f is continuously differentiable on E in Whitney's sense.

If E happens to be a C^1 submanifold of $\mathbf{R}^n$, then f is a C^1 function on E in the usual sense. Otherwise, restrictions of smooth functions on $\mathbf{R}^n$ to E may not be smooth in a more intrinsic way. It may be that E is a non-smooth embedding of a smooth curve or surface, and that smooth functions on $\mathbf{R}^n$ do not correspond to smooth functions on the parameter domain. It may be that E is something like a Lipschitz manifold for which there is not a compatible smooth structure. Nonetheless, we get some regularity to extrinsic smoothness under the geometric condition under consideration.

If E is a chord-arc curve, then one can integrate a continuous family $A(x)$ of differentials on E to get a function f. This is analogous to integrating a continuous function on an interval to get a C^1 function, but it is not quite the same, since the smoothness is now defined extrinsically.

Let us emphasize that this type of geometric condition does not imply anything like Poincaré or Sobolev inequalities, aside from the special case of chord-arc curves. For instance, one can have bubbles with small boundaries where a curve can easily pass. Thus one works with quantities based on the supremum norm instead of integrals.

Just as classical Fourier analysis on the real line deals with the ordinary differential operator d/dx, one can look at analysis on Lipschitz graphs in the complex plane as dealing with certain perturbations of d/dx, as in [4, 5]. The complex derivative of a holomorphic function on a neighborhood of the graph at a point on the graph is a multiple of the ordinary derivative tangent to the graph, which corresponds to a perturbation of d/dx when the graph is parameterized by projecting to the real line. This also works for a complex-valued continuously-differentiable function on the graph in Whitney's sense for which the differentials are complex-linear.

Additional conditions for the differentials of continuously-differentiable functions in Whitney's sense like complex-linearity can be very interesting. In particular, they may imply uniqueness of the differentials. As another version of this, a real-linear mapping from $\mathbf{R}^n$ into a Clifford algebra is uniquely determined by its restriction to any hyperplane when it is left or right Clifford holomorphic. Thus it is natural to look at Clifford-valued continuously-differentiable functions on hypersurfaces in $\mathbf{R}^n$ for which the differentials are left or right Clifford holomorphic. Note that the differentials of a continuously-differentiable function in Whitney's sense are uniquely determined by the function at any point where the set is not approximately contained in a hyperplane.

Normally, the differentials of a continuously-differentiable function in Whitney's sense are not uniquely determined by the function on the set, and they may not be very stable even if they are uniquely determined. The set may be contained in a smooth submanifold of lower dimension, so that the behavior of the differentials in the normal directions is not important. If there is suitable uniqueness and stability, then continuity properties of the differentials can be derived from the corresponding approximation properties of the function by affine functions on the set.

Uniqueness of the differential, perhaps through an additional condition like complex-linearity, has the effect of allowing a derivative in any direction to be interpreted as being tangent to the set. In this way, extrinsic behavior becomes more intrinsic.

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